Problem Categories for this Meet (in addition to topics of earlier meets):

1. Mystery: Problem solving
2. Geometry: Solid Geometry (Volume and Surface Area)
3. Number Theory: Set Theory and Venn Diagrams
4. Arithmetic: Combinatorics and Probability
5. **Algebra: Solving Quadratics with Rational Solutions, including word problems**
Important things you need to know about **ALGEBRA:**

*Solving quadratics with rational solutions, including word problems*

- If \( xy = 0 \), then \( x = 0 \) or \( y = 0 \). This is called the Zero Product Property.

- If \((x - 3)(x + 2) = 0\), then \(x - 3 = 0 \) or \(x + 2 = 0\). The solutions to this problem are \( x = 3 \) and \( x = -2\).

- When a graph crosses the \( x \)-axis, \( y = 0 \).

- To multiply binomials, such as \((x - 4)(x + 6)\), we can use the distributive property. A mnemonic is **FOIL**. Foil means multiply the **F**irst, **O**utside, **I**nside, and **L**ast Terms.

\[
(x - 4)(x + 6) = x^2 + 6x - 4x - 24 = x^2 + 2x - 24
\]

- You should notice that in the above example, the -4 and 6 add to equal 2 and multiply to equal -24. Use this knowledge to work backward to factor a trinomial.

- Factor \( x^2 - 7x + 12 \) (Think: What are two numbers that multiply to equal 12 and add to equal -7? -3 and -4) So, \( x^2 - 7x + 12 = (x - 3)(x - 4) \)

- If \( x^2 - 7x + 12 = 0 \), then \((x - 3)(x - 4) = 0\), so \( x = 3 \) or \( x = 4 \).

- The quadratic formula can also be used to solve quadratic equations. If \( Ax^2 + Bx + C = 0 \), then

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

**Example:** How many units are in the shortest length of the right triangle below?

By Pythagorean Theorem, \( x^2 + (2x + 2)^2 = (2x + 3)^2 \)

So, \( x^2 + (2x + 2)(2x + 2) = (2x + 3)(2x + 3) \)

Using FOIL, \( x^2 + 4x^2 + 4x + 4x + 4 = 4x^2 + 6x = 6x + 9 \)

Combining like terms, \( 5x^2 + 8x + 4 = 4x^2 + 12x + 9 \)

Subtracting the right side to get everything on the left, we get \( x^2 - 4x - 5 = 0 \)

So \((x - 5)(x + 1) = 0\), Therefore, \( x = 5 \) or \( x = -1 \)

A side length cannot be negative, so \( x \) must be 5.

✓ Check. If \( x = 5 \), \( 2x + 2 = 12 \) and \( 2x + 3 = 13 \). \( 5^2 + 12^2 = 13^2 \)
1) If \(( X + 7 ) ( 5X - 2 ) = a^2 + bx + c\), then what is the value of \(a + b + c\)?

\[
\begin{align*}
\text{ANSWERS} \\
1) & \quad \_ \_ \_ \_ \\
2) & \quad \_ \_ \_ \_ \\
3) & \quad \_ \_ \_ \_ \\
\end{align*}
\]
Solutions to Category 5
Algebra
Meet #5 - April, 2017

1) Multiply the two binomials to determine the values of $a$, $b$, and $c$:

$$(x+7)(5x-2) = 5x^2 + 35x - 2x - 14 = 5x^2 + 33x - 14$$

So, $a = 5$, $b = 33$, and $c = -14$.

$a + b + c = 5 + 33 + (-14) = 24$

2) First find the value of $a$ (the coefficient of the first term) by using the data that the two x-intercepts yield the factors $(x + 2)(x - 6)$ and then substituting the value of a third known point for $x$ and $y$:

$y = a(x + 2)(x - 6)$

$-36 = a(0 + 2)(0 - 6)$

$-36 = -12a$

$a = 3$

Now substitute 10 for $x$: $y = 3(10 + 2)(10 - 6)$, so $y = 144$.

3) $H = -4.9t^2 + vt + h$ the given formula

$H = -4.9t^2 + 19.6t + 58.8$ Substitute 19.6 for $v$ and 58.8 for $h$.

$0 = -4.9t^2 + 19.6t + 58.8$ Set the equation to zero, the height of the water.

$0 = t^2 - 4t - 12$ Divide both sides by -4.9.

$0 = (t - 6)(t + 2)$ Factor.

$t = -2$ or $t = 6$ Solve for $t$.

$t = -2$ is extraneous, as it occurred prior to the launch, so we use $t = 6$.

Therefore, it took 6 seconds for the bowling ball to hit the water.
1) If \((N + 3)(N - 7) = 0\), then what is the average of the two possible values of \(N\) that make this quadratic equation true?

2) The quadratic equation \(Y = ax^2 + bx + c\), when graphed, is a parabola that passes through the points (3, 7) and (-5, 15) and has a Y-intercept of (0, -20), as shown. What is the value of \(a + b + c\)?

3) A rocket is launched vertically from ground level at an initial velocity (starting speed) of 128 feet per second. For how many seconds is the rocket at least 112 feet above ground level? Use the quadratic equation \(y = gt^2 + vt + h\) where \(g = -16\) feet/second/second, the constant of gravity at the surface of the Earth, \(t\) is the time in seconds that the rocket is in the air, \(v\) is the initial velocity, \(h\) is the initial height of the rocket in feet, and \(y\) is the height in feet of the rocket at any time \(t\) seconds.

---

**ANSWERS**

1) ________
2) ________
3) ________

Jonas Salk announced his development of the polio vaccine on March 26, 1953 . . . "It is always with excitement that I wake up in the morning, wondering what my intuition will toss up to me, like gifts from the sea. I work with it and rely on it. It's my partner."
1) If \((N + 3)(N - 7) = 0\), then either \(N + 3 = 0\) or \(N - 7 = 0\), then \(N = -3\) or \(N = 7\). The average of these solutions is \((-3 + 7) / 2\), or 2.

2) One possible strategy:
   1: substitute the X and Y coordinates of each of the known points into the general quadratic equation,
   2: solve the resulting system to find the values of a, b, and c, and then
   3: find the sum \(a + b + c\).

for \((3, 7)\): \[7 = a(3^2) + b(3) + c\] ... or, simplified, \[7 = 9a + 3b + c\]

for \((-5, 15)\): \[15 = a((-5)^2) + b(-5) + c\] ... or, simplified, \[15 = 25a - 5b + c\]

for \((0, -20)\): \[-20 = a(0^2) + b(0) + c\] ... or, simplified, \[-20 = c\].

Now substitute \(-20\) for \(c\) in the first two equations, yielding \[7 = 9a + 3b - 20\] and \[15 = 25a - 5b - 20\]

Simplifying: \[27 = 9a + 3b\] and \[35 = 25a - 5b\]

Divide both sides of the first equation by 3 and both sides of the second equation by 5, yielding: \[9 = 3a + b\] and \[7 = 5a - b\]. Adding the two equations yields: \[16 = 8a\], so, \(a = 2\) and then \(b = 3\). So, \(a + b + c = -15\).

3) Substitute: \(G = -16;\) \(V = 128;\) \(Y = 112;\) \(H = 0\).

\[112 = (-16)(T^2) + 128T + 0\] Use the substitutions listed above.

\[0 = -16(T^2) + 128T - 112\] Subtract 112 from both sides.

\[0 = T^2 - 8T + 7\] Divide both sides by -16.

\[0 = (T - 1)(T - 7)\] Factor.

\(T = 1\) or \(T = 7\). Therefore, the rocket was at or above 112 feet above the ground from 1 second until 7 seconds into the flight, so the rocket was in flight for the difference 7 - 1, or 6 seconds.
1. Terry is trying to graph a particular quadratic equation on a coordinate system. She found that the vertex of the parabola is at the point \((-2.5, 90.25)\) and that the parabola crosses the \(x\)-axis at \((7, 0)\). Find the coordinates of the other point where the parabola crosses the \(x\)-axis.

\[\text{Answers}\]

1. \((______,______)\)
2. ______________
3. _______ sq. units

2. The sum of a number and its reciprocal is \(2\frac{1}{30}\). What is the number if it is less than 1?

3. Triangle ABC below is a right triangle with a right angle at vertex A. How many square units are there in the area triangle ABC?
Solutions to Category 5
Algebra
Meet #5, March/April 2013

1. We should not be put off by the large $y$ value of the vertex. The important part is the $x$ value of the vertex, which is $-2.5$. The parabola has a vertical line of symmetry where $x = -2.5$. We know that one root of the equation is at $(7, 0)$, which is 9.5 units to the right of $(-2.5, 0)$. The other root must be 9.5 units to the left of $(-2.5, 0)$, which is $(-12, 0)$.

Note: The points where the parabola crosses the $x$ axis are also known as the “roots” of the equation.

2. If we call the unknown number $x$ and its reciprocal $1/x$, we get the equation $x + \frac{1}{x} = 2 \frac{1}{30}$, which is equivalent to $x + \frac{1}{x} = \frac{61}{30}$. If we multiply both sides of the equation by $30x$, we get $30x^2 + 30 = 61x$. This is a quadratic equation, so we set it equal to zero as: $30x^2 - 61x + 30 = 0$. This is not easily factored, but can be solved using the quadratic formula.

A more experienced mathlete might call the unknown number $a/b$ and its reciprocal $b/a$. The sum is thus $\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = \frac{a^2 + b^2}{ab}$, which reveals much more about the structure of the problem. Our sum is $61/30$, so we need to find two factors of 30 such that the sum of their squares is 61. The obvious numbers to try are 5 and 6, and indeed $\frac{5}{6} + \frac{6}{5} = \frac{25}{30} + \frac{36}{30} = \frac{61}{30}$. The number must be $\frac{5}{6}$.

3. We can use the Pythagorean Theorem to solve for $x$ as shown at right. The value zero doesn’t make sense, so $x$ must be 10 units. The length of the other leg must be $2 \times 10 + 4 = 24$ units. (The hypotenuse is $3 \times 10 - 4 = 26$ units.) The area of the triangle is thus $10 \times 24 \div 2 = 120$ square units.

Answers
1. $(-12, 0)$
2. $\frac{5}{6}$
3. 120 sq. units
Category 5 – Algebra

1. If we add 4 inches to a square’s length, and shorten its width by 4 inches, we get a new rectangle whose area is 75% of the original square’s area. How many inches are in the length of the square’s side?

2. One of the solutions to the equation $x^2 + B \cdot x - 8 = 0$ is $x = 2$. What is the value of the other solution?

3. The product of the solutions of the equation $x^2 - 9x + C = 0$, is twice as much as their sum. What is the larger of the two solutions?

Answers

1. _______________

2. _______________

3. _______________
Solutions to Category 5 - Algebra

1. If we call the square’s length $x$, then we can write:

$$(x + 4) \cdot (x - 4) = x^2 - 16 = \frac{3}{4} \cdot x^2$$

and so $x^2 = 64$ and $x = 8$ inches. *Since we’re looking for a length, we’re interested only in the positive solution.*

2. Since we know that $x = 2$ is a solution, we can use this value in the original equation to get $2^2 + B \cdot 2 - 8 = 0$ and solve to get $B = 2$.

Now the original equation is known to be $x^2 + 2 \cdot x - 8 = 0$ and the other solution for this is $x = -4$.

3. In the general case of a quadratic equation $A \cdot x^2 + B \cdot x + C = 0$ we know that the sum of solutions is $-B$, and their product is $A \cdot C$. In our case $A = 1$ and $B = -9$, and so the sum of solutions is $+9$, and their product is $18 = C$.

Our equation then is $x^2 - 9 \cdot x + 18 = (x - 3) \cdot (x - 6) = 0$.

The solutions are of course $x = 3$ and $x = 6$. 

Answers

1. 8
2. −4
3. 6
Category 5
Algebra
Meet #5, March 2009

1. What is the positive difference between the two solutions to the equation below?

\[ 3x^2 + 12x - 31 = 32 \]

2. The diagram below is a large rectangle which is made up of a square and 2 rectangles as shown. The area of the entire figure is 756. What is the area of the square?

![Diagram]

3. The difference between a positive number and \(3 \frac{1}{3}\) times its reciprocal is equal to \(1 \frac{1}{6}\). What is the number? Express your answer as a common fraction.

Answers
1. _______________
2. _______________
3. _______________
Solutions to Category 5
Algebra
Meet #5, March 2009

Answers

1.  
   \[3x^2 + 12x - 31 = 32\]
   \[3x^2 + 12x - 63 = 0\]
   Dividing both sides of the equation by 3 gives us:
   \[x^2 + 4x - 21 = 0\]
\(x + 7 = 0\) or \(x - 3 = 0\)
\[x = -7\text{ or } x = 3\]
The positive difference between two solutions is \(3 - (-7) = 10\)

2. If we call the side of the square \(x\), the square has area \(x^2\), the rectangle to the right of the square has area \(3x\) and the area of the rectangle below the square is \(4(x + 3) = 4x + 12\).
The total area of the three shapes is \(x^2 + 3x + 4x + 12 = x^2 + 7x + 12 = 756\).
\(x^2 + 7x - 744 = 0\)
\((x + 31)(x - 24) = 0\)
x + 31 = 0 or x - 24 = 0
\[x = -31\text{ or } x = 24\]
Since the side length of the square cannot be negative, the side of the square is 24 and the area of the square is \(24^2 = 576\).

3. \[x - 3 \frac{1}{3} \left(\frac{1}{x}\right) = 1 \frac{1}{6}\]
   \[x - \frac{10}{3} \left(\frac{1}{x}\right) = \frac{7}{6}\]
   \[x - \frac{10}{3x} = \frac{7}{6}\]
Multiplying both sides of the equation by \(6x\) gives us:
\[6x^2 - 20 = 7x\]
\[6x^2 - 7x - 20 = 0\]
\((2x - 5)(3x + 4) = 0\)
\[2x - 5 = 0\text{ or } 3x + 4 = 0\]
\[2x = 5\text{ or } 3x = -4\]
\[x = \frac{5}{2}\text{ or } x = -\frac{4}{3}\]
Since we know \(x\) is positive, it must be \(\frac{5}{2}\).
Category 5  
Algebra  
Meet #5, March 2007

1. Give the lesser of the two solutions to the following equation.

\[ x(7 + x) + 312 - 4x = 3(x + 137) + 526 \]

2. The length and width of a certain rectangle are whole numbers of units, and the length is one more than four times the width. If the area of the rectangle is 105 square units, how many units are in the perimeter of the rectangle?

3. The graph of the quadratic equation shown at right crosses the x-axis at the points (-5, 0) and (7, 0), and its vertex is at the point (1, -36). Give the coordinates of the two points where the graph of this same equation crosses the line \( y = 13 \). Reminder: The coordinates of the two points should be written as ordered pairs in parentheses.

<table>
<thead>
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<td>1. ____________</td>
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<tr>
<td>2. ____________</td>
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<td>3. _______ and _______</td>
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Solutions to Category 5
Algebra
Meet #5, March 2007

Answers

1. The equation can be simplified as follows:
   \[ x(7 + x) + 312 - 4x = 3(x + 137) + 526 \]
   \[ 7x + x^2 + 312 - 4x = 3x + 411 + 526 \]
2. 52
   \[ x^2 + 3x + 312 = 3x + 937 \]
3. (8, 13) and 
   \( (-6, 13) \)
   \[ x^2 + 312 = 937 \]
   \[ x^2 = 625 \]
   The two solutions are \( x = 25 \) and \( x = -25 \). We want \(-25\).

2. If we call the width \( x \), then the length is \( 4x + 1 \). Since length times width gives the area of a rectangle, we can write the quadratic equation \( x(4x + 1) = 105 \).

Distributing on the left, we can rewrite this as \( 4x^2 + x = 105 \). Now we subtract 105 from each side to set the equation equal to zero: \( 4x^2 + x - 105 = 0 \). At this point we can use the quadratic formula or try to factor the trinomial into a product of two binomials. Our template for factoring is: \( (\_x + \_)(\_x - \_) = 0 \).

The prime factorization of 4 is \( 2 \times 2 \) and that of 105 is \( 3 \times 5 \times 7 \). We need to place these factors in the blanks so that the difference is 1 for the middle term. We note that \( 2 \times 2 \times 5 = 20 \) and \( 3 \times 7 = 21 \), so our equation must be \( (4x + 21)(x - 5) = 0 \).

The two solutions are \( x = -\frac{21}{4} \) and \( x = 5 \). Only 5 units make sense for the width, so the length must be \( 4 \times 5 + 1 = 21 \) units. The perimeter of the rectangle is \( 2 \times (5 + 21) = 2 \times 26 = 52 \) units.

3. We are told that the equation is quadratic and we are given the roots of the equation. We can assume that the equation has the form \( y = a(x + 5)(x - 7) \), where \( a \) is not yet determined. We can use the vertex, \( (1, -36) \), as a point to help us determine the value of \( a \). First we substitute \( y = -36 \) and \( x = 1 \) into the equation to get \( -36 = a(1 + 5)(1 - 7) \). Now we simplify and get \( -36 = -36a \), which shows us that \( a = 1 \). Now we want to find the values of \( x \) when \( y = 13 \). We have to solve the equation \( 13 = 1(x + 5)(x - 7) \). Expanding on the right, we get
   \[ 13 = x^2 - 2x - 35 \]. Now we set this equal to zero and factor the new equation as follows: \( 0 = x^2 - 2x - 48 = (x - 8)(x + 6) \). From this, we find that the desired \( x \) values are 8 and \(-6\). The coordinates of the two points where the graph crosses the line \( y = 13 \) are \( (8, 13) \) and \( (-6, 13) \).