Math League SCASD
2019-20

Meet #1

Number Theory

Self-study Packet

Problem Categories for this Meet:

1. Mystery: Problem solving
2. Geometry: Angle measures in plane figures including supplements and complements
3. Number Theory: Divisibility rules, factors, primes, composites
4. Arithmetic: Order of operations; mean, median, mode; rounding; statistics
5. Algebra: Simplifying and evaluating expressions; solving equations with 1 unknown including identities
**Information you need to know about NUMBER THEORY...**

**Divisibility Rules**

A number is divisible by:
- ♦ 2 if its ones digit is even (0, 2, 4, 6, 8)
- ♦ 3 if the sum of its digits is divisible by 3 (for example, to check if 364 is divisible by 3, add 3 + 6 + 4. You get 13. 13 is not divisible by 3, so 364 is not either).
- ♦ 4 if the number formed by its last two digits is divisible by 4 (for example, to check if 2,320 is divisible by 4, look at the number formed by the last two digits; in this case, 20. 20 is divisible by 4, so 2,320 is as well).
- ♦ 5 if its ones digit is 0 or 5
- ♦ 6 if it divisible by 2 and 3
- ♦ 8 if the number formed by its last three digits is divisible by 8 (similar to the rule for 4)
- ♦ 9 if the sum of its digits is divisible by 9 (similar to the rule for 3)
- ♦ 10 if its ones digit is 0

**Factoring**

To find the factors of a number, it is useful to find the pairs of numbers that multiply to give you that number. Using the divisibility tricks, it is much easier to narrow this down. I recommend starting with 1. You can stop when you pass the square root of the number, because you have found all the factors.

For example, list all the factors of 120.

\[
\begin{array}{cccc}
1 \times 120 & 2 \times 60 & 3 \times 40 & 4 \times 30 & 5 \times 24 \\
6 \times 20 & 8 \times 15 & 10 \times 12 \\
\end{array}
\]

The factors of 120 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120.

**Primes and Composites**

**Prime Number:** any number with exactly two factors

**Composite Number:** any number with more than two factors

**0 and 1 are neither prime nor composite!**
Category 3
Number Theory
Meet #1 - October, 2017

1) Find the sum of all composite numbers between 37 and 47.

2) What is the smallest number that is divisible by five different prime numbers?

3) is divisible by 3.
   has an odd number of factors.
   is divisible by 4.

What is the sum of all possible values of between 1 and 1000?

Answers
1) 
2) 
3) 


Solutions to Category 3
Number Theory
Meet #1 - October, 2017

1) The composites between 37 and 47 are 38, 39, 40, 42, 44, 45, and 46. The only primes in that range are 41 and 43. The sum of the composites is $38 + 39 + 40 + 42 + 44 + 45 + 46 = 294$.

2) \(2 \times 3 \times 5 \times 7 \times 11 = 2310\).

3) Whole numbers with an odd number of factors are perfect squares. Those between 1 and 1000 that are also divisible by 4 and 3 (multiples of 12) are the following:

\[
\begin{align*}
36 &= 4 \times 9 \\
144 &= 4 \times 4 \times 9 \\
324 &= 4 \times 9 \times 9 \\
576 &= 8 \times 8 \times 9 \\
900 &= 4 \times 9 \times 25 \\
\end{align*}
\]

\[
36 + 144 + 324 + 576 + 900 = 1980
\]
1) Find the sum of all composite numbers between 47 and 59.

2) If $A$, $B$, and $C$ are prime numbers, and $A > B > C$, then how many different factors does the product $ABC$ have?

3) A two-digit whole number is divisible by $E$ and its units digit (the number in the ones place) is also $E$. The two digits are different. What is the largest possible value of $E$?

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October 31, 1941 - Mount Rushmore National Memorial was completed after 14 years of work. The memorial contains 60-foot-tall sculptures of the heads of Presidents George Washington, Thomas Jefferson, Abraham Lincoln and Theodore Roosevelt - representing America's founding, political philosophy, preservation, and expansion and conservation.
1) Composite numbers, unlike prime numbers, have at least three factors.
   \[ 48 + 49 + 50 + 51 + 52 + 54 + 55 + 56 + 57 + 58 = 530. \]

2) Select any three prime numbers - A, B, and C - multiple them, and the product will have eight factors:
   1, A, B, C, AB, AC, BC, and ABC.

3) The two-digit numbers where the two digits are different and the number is divisible by its units digit are:
   12, 15, 21, 24, 25, 31, 32, 35, 36, 41, 42, 45, 48, 51, 52, 61, 62, 63, 64, 65, 71, 72, 75, 81, 82, 84, 85, 91, 93, 95, and 96.
   Therefore, the largest possible value of E comes compliments of the number 48 and is therefore 8.
It would be VERY useful to memorize all the prime numbers under 100 (Questions where you need to know them occur over and over again all season). They are:

2  3  5  7  11  13  17
19  23  29  31  37  41  43
47  53  59  61  67  71  73
79  83  89  97

**Prime Factoring:** Two common ways to find the prime factorization of a number are to use a factor tree or to use the ladder method. An example of each follows:

**Factor Tree:** Find the prime factorization of 200.

1. Start with your original number.
2. Find a pair of numbers that multiply to give you your original number.
3. Find a pair of numbers that multiply to give you each of the factors.
4. Continue until you only have prime numbers.
5. The prime factorization is written as a product of all the primes.

![Factor Tree Diagram]

200 = 2³ × 5²

**Ladder Method:** Find the prime factorization of 60.

1. Start with the lowest prime number (2) and check if your number is divisible by it.
2. If it is, divide your number by that. If not, try 3, then 5, then 7, etc, until you find a number that is a factor of your original number. Divide by it.
3. Repeat by dividing your new number by its smallest prime factor.
4. Continue until you are left with 1.
5. The prime factorization is written as a product of all its primes, which are conveniently ordered on the left of your ladder.

\[
\begin{array}{c|c}
| & 60 \\
\hline
2 & 30 \\
\hline
3 & 15 \\
\hline
5 & 5 \\
\hline
1 & \\
\end{array}
\]

60 = 2² × 3 × 5
Category 3  
Number Theory  
Meet #1 - October, 2013

1) What is the only whole number between 280 and 290 that is divisible by both 4 and 9?

2) Two of the four prime factors of 17,017 are 13 and 17. What is the larger of the other two prime factors?

3) □ is a multiple of 9.  
□ > 1000.  
□ is divisible by both 5 and 7.  
□ is not divisible by 2.  
□ < 2000.  
What is the value of □?

Answers
1) □□□□
2) □□□□
3) □□□□
Solutions to Category 3
Number Theory
Meet #1 - October, 2013

<table>
<thead>
<tr>
<th>Answers</th>
<th>1) Since 4 and 9 are relatively prime, their product is their LCM. We need to look at multiples of 36 that lie near 280. A bit of guessing and checking yields (36)(8) = 288.</th>
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<tbody>
<tr>
<td></td>
<td>1) 288</td>
</tr>
<tr>
<td></td>
<td>2) 11</td>
</tr>
<tr>
<td></td>
<td>3) 1575</td>
</tr>
</tbody>
</table>

2) First, divide 17,017 by the product of 13 and 17. Then factor the resulting quotient by using divisibility rules:

(13)(17) = 221.
17,017 / 221 = 77.
77 = (7)(11).
The larger of the two factors of 77 is 11.

3) We are looking for an odd multiple of (5)(7)(9) that lies between 1000 and 2000.

(5)(7)(9) = 315.
The multiples of 315 that lie between 1000 and 2000 are
4(315) = 1260,
5(315) = 1575, and
6(315) = 1890.
The only odd multiple is 1575.
category 3 – number theory

1. what is the smallest natural number which is divisible by 3, 4, 5, 6, and 7?

2. what is the smallest 5-digit number that is a multiple of 18?

3. what is the sum of all factors of the number 1,000? (including 1 and itself).

answers

1. ______________

2. ______________

3. ______________
1. In order to be divisible by 3 and 4, a number has to be a multiple of $3 \cdot 4 = 12$. Adding the requirements for 5 and 7, we have to multiply by $5 \cdot 7 = 35$, so we get $12 \cdot 35 = 420$. We don’t have to worry about the 6, as any multiple of 12 is sure to be a multiple of 6.

2. The answer is one of the numbers in the range $\{10,000 - 10,017\}$. In order to be divisible by 18, our number has be divisible by both 2 and 9, so we’re looking for an even number with a sum of digits that divides by 9, and the smallest such number possible is $10,008 = 18 \times 556$.

3. $1,000 = 1 \times 1,000 = 2 \times 500 = 4 \times 250 = 5 \times 200 = 8 \times 125 = 10 \times 100 = 20 \times 50 = 25 \times 40$

   The sum of factors then is:
   
   $1 + 2 + 4 + 5 + 8 + 10 + 20 + 25 + 40 + 50 + 100 + 125 + 200 + 250 + 500 + 1000 = 2,340$

Answers

1. 420
2. 10008
3. 2340
1. What is the positive difference between the sum of the 6 largest primes that are less than 20 and the sum of the 3 smallest composites that are greater than 30?

2. The number \(1X38X\) is divisible by 12 when \(X = A\), and by 9 when \(X = B\). What is the value of \(A+B\)?

3. If \(S_{66}\) represents the sum of all the factors of the number 66 and \(S_{70}\) represents the sum of all the factors of the number 70, then find the value of \(S_{70} - S_{66}\).
1. The first primes are 2, 3, 5, 7, 11, 13, 17, 19,… so the sum of the 6 largest under 20 is
\[ 5 + 7 + 11 + 13 + 17 + 19 = 72. \]
The first composites greater than 30 are 32, 33, 34 and their sum is 99.
The positive difference is \( 99 - 72 = 27 \).

2. \( 1A38A \) has to be divisible by 12 and so both by 3 and by 4.
   Divisibility by 3 means that the sum of digits \( (2 \cdot A + 12) \) is a multiple of 3, which
   means \( A \) can be one of the digits \([0, 3, 6, 9]\). Divisibility by 4 means that the number
   \( 8A \) has to be divisible by 4, so \( A \) can be one of the digits \([0, 4, 8]\). The only digit to
   match both criteria is \( A = 0 \).
   \( 1B38B \) is divisible by 9, so its sum of digits has to be divisible by 9. That means that
   \( (2 \cdot B + 12) \) is a multiple of 9, and the only possible value for the digit \( B \) is 3.
   Therefore \( A + B = 0 + 3 = 3 \).

3. \( S_{66} = 1 + 2 + 3 + 6 + 11 + 22 + 33 + 66 = 144 \)
   \( S_{70} = 1 + 2 + 5 + 7 + 10 + 14 + 35 + 70 = 144 \)
   \( S_{70} - S_{66} = 0 \)
1. Billy says the divisibility rule for 60 is that you check and see if the number is divisible by 6 and 10. Sally says that he is wrong (and Sally is correct). Sally says that the best way to check if a number is divisible by 60 is to see if it is divisible by X, Y, and Z with X, Y, and Z all greater than 1. What is the minimum value of Y \cdot (X+Z)?

2. If P represents the sum of the prime numbers between 40 and 60, and C represents the sum of the odd composite numbers between 40 and 60, what is the positive difference between P and C?

3. How many factors of 432 are the squares of positive integers?

Answers
1. _______________
2. _______________
3. _______________
1. Billy is wrong because numbers like 30 are divisible by 6 and 10, but not by 60. Since 60 is equal to $3 \cdot 4 \cdot 5$ and 3, 4, and 5 are relatively prime, any number divisible by 60 must be divisible by 3, 4, and 5. Therefore $X$, $Y$, and $Z$ are equal to 3, 4, and 5 in some order. Checking the three possible combinations we can find that the minimum value of $Y\cdot(X+Z)$ is then $3(4 + 5) = 3(9) = 27$.

2. 

$$
P = 41 + 43 + 47 + 53 + 59 = 243
$$
$$
C = 45 + 49 + 51 + 55 + 57 = 257
$$

$$
C - P = 257 - 243 = 14
$$

3. You could list out all the factors and pick out the squares:

1. 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 36, 48, 54, 72, 108, 144, 216, 432

So there are 6 factors of 432 that are perfect squares.

**For a more elegant solution**

You could look at the ways a perfect square might be made from factors by looking at the prime factorization. (*note: prime factorization is not necessary for this problem as shown first, this is just an alternate solution*)

$$
432 = 2^4 \times 3^3
$$

Since square numbers have an even number of the same factors multiplied together, a perfect square factor of 432 could be divisible by $2^0$, $2^2$, or $2^4$. It could also be divisible by $3^0$ or $3^2$. That gives us 3 choices for divisibility by two and 2 choices for divisibility by three. That gives us a total of $3 \times 2 = 6$ square factors of 432.